(5) Resolva a EDO $xy'' + 2y' = 12x^2$ fazendo a substituição $u = y'$.

$$u = y' \Rightarrow u' = y''$$

$$\therefore \; xy'' + 2y' = 12x^2 \Rightarrow xu' + 2u = 12x^2$$

$(\div x)$

$$\Rightarrow u' + \frac{2}{x}u = 12x \quad \text{(linear)}$$

Fator integrante: $e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln x} = e^{\ln(x^2)}$

$$= x^2$$

$$\left(e^{2\ln x} = e^{\ln x + \ln x} = e^{\ln x} \cdot e^{\ln x} = x \cdot x = x^2 \right)$$

$$\therefore \; x^2\left(u' + \frac{2}{x}u\right) = x^2 \cdot 12x \Rightarrow x^2u' + 2xu = 12x^3$$

$$\Rightarrow (x^2 \cdot u)' = 12x^3 \Rightarrow \int (x^2 \cdot u)' dx = \int 12x^3 dx$$

$$\Rightarrow x^2 \cdot u = 3x^4 + C \Rightarrow u = \frac{3x^4 + C}{x^2} = 3x^2 + \frac{C}{x^2}$$

Assim,

$$y' = 3x^2 + \frac{C}{x^2} \Rightarrow \int y' dx = \int 3x^2 + \frac{C}{x^2} dx$$

$$\Rightarrow y + C_1 = x^3 - \frac{C}{x} + C_2 \Rightarrow \boxed{y = x^3 - \frac{C}{x} + K}$$

$$\int x^{-2} dx = \frac{x^{-1}}{-1} + C$$

Eq. linear de 2ª ordem

$$P(x)y'' + Q(x)y' + R(x)y = G(x), \quad P, Q, R, G \text{ contínuas}$$

- Eq. Lineares de 2ª ordem com coef. constantes:

$$ay'' + by' + cy = G(x), \quad a, b, c \in \mathbb{R}, \quad a \neq 0, \quad G \text{ contínua}$$

- Eq lineares de 2ª ordem com coef. constantes e homogênea:

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, \quad a \neq 0$$

$$y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$$

$$\therefore a(r^2 e^{rx}) + b(re^{rx}) + ce^{rx} = 0$$

$$(e^{rx} > 0)$$

$$\Rightarrow e^{rx} \cdot (ar^2 + br + c) = 0 \Rightarrow \underbrace{ar^2 + br + c = 0}_{\substack{\text{eq. auxiliar} \\ \text{eq. carcterística}}}$$

Exemplos: 1) $y'' + y' - 6y = 0$

Eq. carac.: $r^2 + r - 6 = 0 \Rightarrow r_1 = -3$ e $r_2 = 2$

$\therefore y_1 = e^{-3x}$ e $y_2 = e^{2x}$ são sol. da EDO.

Verificando:

- $y_1 = e^{-3x} \Rightarrow y_1' = -3 \cdot e^{-3x} \Rightarrow y_1'' = 9e^{-3x}$

$\Rightarrow 9e^{-3x} + (-3)e^{-3x} - 6 \cdot e^{-3x} = 0.$ ✓

- $y_2 = e^{2x} \Rightarrow y_2' = 2e^{2x} \Rightarrow y_2'' = 4e^{2x}$

$\Rightarrow 4e^{2x} + 2e^{2x} - 6 \cdot e^{2x} = 0.$ ✓

Observe que:

$$y_1 + y_2 = e^{-3x} + e^{2x} \Rightarrow (y_1 + y_2)' = -3e^{-3x} + 2e^{2x}$$

$$\Rightarrow (y_1 + y_2)'' = 9e^{-3x} + 4e^{2x}$$

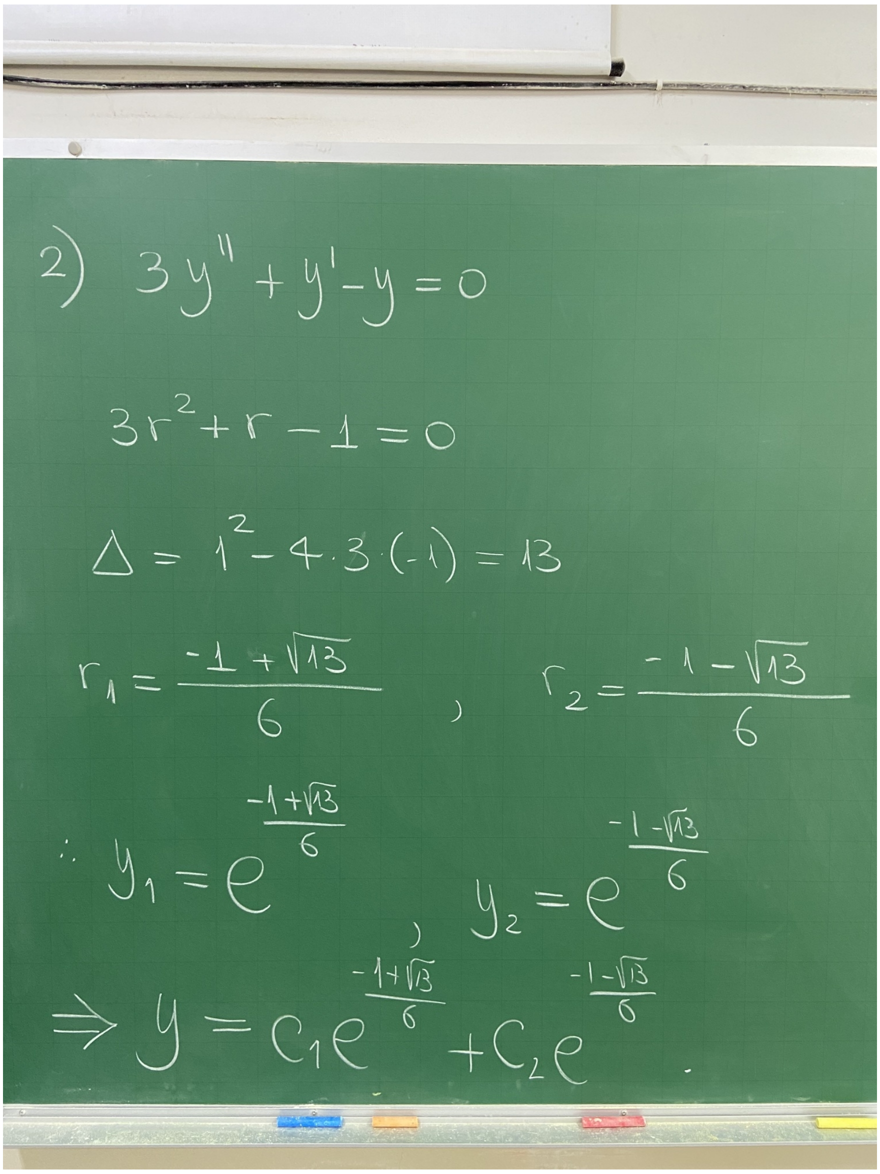
Na eq.:

$$(y_1+y_2)'' + (y_1+y_2)' - 6(y_1+y_2)$$

$$= 9e^{-3x} + 4e^{2x} + (-3)e^{-3x} + 2e^{2x} - 6\left(e^{-3x} + e^{2x}\right) = 0$$

Analogamente, cy_1 e cy_2 também são soluções.
(exercício!)

Portanto, $y = c_1 y_1 + c_2 y_2 = \underbrace{c_1 e^{-3x} + c_2 e^{2x}}_{\text{sol. geral.}}$ é sol.



2) $3y'' + y' - y = 0$

$3r^2 + r - 1 = 0$

$\Delta = 1^2 - 4 \cdot 3 \cdot (-1) = 13$

$r_1 = \dfrac{-1 + \sqrt{13}}{6}$, $r_2 = \dfrac{-1 - \sqrt{13}}{6}$

$\therefore\ y_1 = e^{\frac{-1+\sqrt{13}}{6}}$, $y_2 = e^{\frac{-1-\sqrt{13}}{6}}$

$\Rightarrow y = c_1 e^{\frac{-1+\sqrt{13}}{6}} + c_2 e^{\frac{-1-\sqrt{13}}{6}}$.

3) $4y'' + 12y' + 9y = 0$

$4r^2 + 12r + 9 = 0$

$\Delta = 12^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$. (raiz real com multiplicidade)

$r = \frac{-12}{8} = -\frac{3}{2} \Rightarrow y_1 = e^{-\frac{3}{2}x}$

e $y_2 = xe^{-\frac{3}{2}x}$ ←

$\Rightarrow y = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}$.

4) $y'' - 6y' + 13y = 0$

$a + bi$, $i = \sqrt{-1}$

$\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4i$

$r^2 - 6r + 13 = 0$

$\Delta = (-6)^2 - 4 \cdot 1 \cdot 13 = 36 - 52 = -16$.

$r_1 = \frac{6 + \sqrt{-16}}{2} = \frac{6 + 4i}{2} = 3 + 2i \Rightarrow y_1 = e^{(3+2i)x}$

$r_2 = \frac{6 - \sqrt{-16}}{2} = \frac{6 - 4i}{2} = 3 - 2i \Rightarrow y_2 = e^{(3-2i)x}$

$\Rightarrow y = C_1 e^{(3+2i)x} + C_2 e^{(3-2i)x}$.

Fórmula de Euler: $e^{ix} = \cos x + i\,\mathrm{sen}\,x$.

- $e^{(3+2i)x} = e^{3x+2ix} = e^{3x} \cdot e^{i(2x)}$

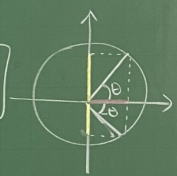
$$= e^{3x}\left[\cos(2x) + i\,\mathrm{sen}(2x)\right]$$

- $e^{(3-2i)x} = \dots = e^{3x}\left[\cos(-2x) + i\,\mathrm{sen}(-2x)\right]$

$$\Rightarrow y = c_1 e^{3x}\left[\cos(2x) + i\,\mathrm{sen}(2x)\right] + c_2 e^{3x}\left[\cos(-2x) + i\,\mathrm{sen}(-2x)\right]$$

$$= e^{3x}\left[c_1\cos(2x) + ic_1\,\mathrm{sen}(2x) + c_2\cos(-2x) + ic_2\,\mathrm{sen}(-2x)\right]$$

$$= e^{3x}\left[c_1\cos(2x) + ic_1\,\mathrm{sen}(2x) + c_2\cos(2x) - ic_2\,\mathrm{sen}(2x)\right]$$



$$= e^{3x}\left[\underbrace{(c_1+c_2)}_{C_1}\cos(2x) + \underbrace{(ic_1-ic_2)}_{C_2}\,\mathrm{sen}(2x)\right]$$

$$= e^{3x}\left[C_1\cos(2x) + C_2\,\mathrm{sen}(2x)\right]$$

Resumindo: $ay'' + by' + cy = 0 \Rightarrow ar^2 + br + c = 0$

- $r_1 \neq r_2$ reais: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- r real com mult.: $y = c_1 e^{rx} + c_2 x e^{rx}$
- $r_1, r_2 \in \mathbb{C}$, $r_1 = \alpha + \beta i$: $y = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \operatorname{sen}(\beta x)\right]$